Iteratively Reweighted Flexible Krylov methods for Sparse Reconstruction

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We are interested in the solution of

$$Ax + e = b, \quad A \in \mathbb{R}^{M\times N}, \quad b \in \mathbb{R}^M, \quad e \in \mathbb{R}^M$$

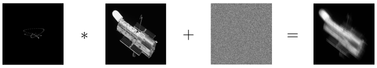
coming from suitable discretization of

$$\int_\Omega k(s,t)f(t)dt + \varepsilon(s) = g(s).$$

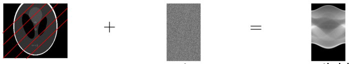
Modeling inverse problems:

- the process k (resp. A), the output g (resp. b) are known;
- the input f (resp. x) is unknown.

Image deblurring and denoising.



Tomography reconstruction and denoising.



Iterative Regularization: Krylov Subspace Methods

[Saad. Iterative Methods for Sparse Linear Systems. *SIAM*, 2003]

Common framework for Krylov Subspace methods for

$$\arg\min_{x\in\mathbb{R}^N} \|Ax - b\|_2 .$$

At step k:

- Expand the solution (Krylov) subspace:

$$\mathcal{R}(V_k) = \mathcal{K}_k(C,d) = \text{span}\{d, Cd, \dots, C^{k-1}d\},$$

  by updating a partial decomposition of A of the form:

$$AV_k = U_{k+1}H_k , \quad \text{with} \quad H_k \in \mathbb{R}^{(k+1)\times k}.$$

- Consider the approximation $x_k = V_k y_k$.
- Solve the projected LS:

$$y_k = \arg\min_{y\in\mathbb{R}^k} \|b - AV_k y\|_2 = \arg\min_{y\in\mathbb{R}^k} \|d_k - H_k y\|_2$$

Sparsity enforcing ℓ_p regularisation term

[P. Rodriguez and B. Wohlberg, An efficient algorithm for sparse representations with ℓ^p data term. In proc. of the 4th IEEE Andean Technical Conf. (AN-DESCON), 2008.]

For $\lambda, p > 0$

$$\arg\min_{x\in\mathbb{R}^N} \|Ax - b\|_2^2 + \lambda\|x\|_p^p = \arg\min_{x\in\mathbb{R}^N} \|Ax - b\|_2^2 + \lambda\left\|W^{(p)}(x)\,x\right\|_2^2$$

$$W^{(p)}(x) = \text{diag}\left((|[x]_i|^{\frac{p-2}{2}})_{i=1,\dots,n} \right)$$

Approximate by a **sequence**:

$$x_k = \arg\min_{x\in\mathbb{R}^N} \|Ax - b\|_2^2 + \lambda\,\|W_k\,x\|_2^2 + c_k, \quad W_k = W^{(p)}(x_{k-1})$$

or, equivalently,

$$\bar{x}_k = \arg\min_{\bar{x}} \|AW_k^{-1}\bar{x} - b\|_2^2 + \lambda\|\bar{x}\|_2^2, \quad \text{so} \quad x_k = W_k^{-1}\bar{x}_k.$$

Iterative Regularization: FGMRES and FLSQR

[Saad. A flexible prec. GMRES. *SISC*, 1993.]

[Chung&Gazzola. Flexible Krylov Methods for ℓ_p regularization. SIAM J.Sci.Comput., 2019]

At step k:

- Expand a search (Flexible Krylov) subspace:
  $\mathcal{R}(Z_k) = \text{span}\{W_1^{-1}v1, W_2^{-1}v_2, W_3^{-1}v_3, \dots\}$
  expanding the decomposition

$$AZ_k = V_{k+1}H_k \text{, with } H_k \in \mathbb{R}^{(k+1)\times k}$$

- Consider the approximation $x_k = Z_k y_k$.
- Solve the projected LS:

$$y_k = \arg\min_{s\in\mathbb{R}^k} \|b - AZ_k s\|_2 = \arg\min_{s\in\mathbb{R}^k} \|d_k - H_k y\|_2.$$

$$\min_{x\in\mathbb{R}^N} \|Ax - b\|_2^2$$

$$\min_{x\in\mathbb{R}^N} \|Ax - b\|_2^2 + \lambda\,\|W_k x\|_2^2$$

$$\min_{y\in\mathbb{R}^k} \|AZ_k y - b\|_2^2 + \lambda\,\|W_k Z_k y\|_2^2$$

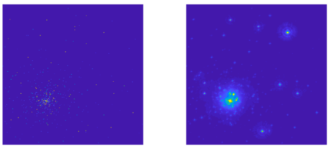


Fig. 1: Setting for the star_cluster test problem. (a) True image x_{true}, (b) Noisy measurement b.

Define

$$T^{(p)}(x) = \|Ax - b\|_2^2 + \lambda\left\|\widetilde{W}^{(p)}(x)\,x\right\|_2^2$$

Lemma The sequence $\{T^{(p)}(x_k)\}_{k\geq 1}$ for $0 < p \leq 2$, where x_k is the approximated solution computed after k steps of the IRW-FGMRES or the IRW-FLSQR methods, is decreasing monotonically and it is bounded from below by zero.

Theorem The sequence $\{x_k\}_{k\geq 1}$, where x_k is the approximated solution computed after k steps of IRW-FGMRES or IRW-LSQR with $p > 0$, is such that

$$\lim_{k\to\infty} \|x_k - x_{k-1}\|_2 = 0.$$

Moreover, it converges to a stationary point of $T^{(p)}$ and, if $p \geq 1$, this is the unique minimizer of $T^{(p)}$.

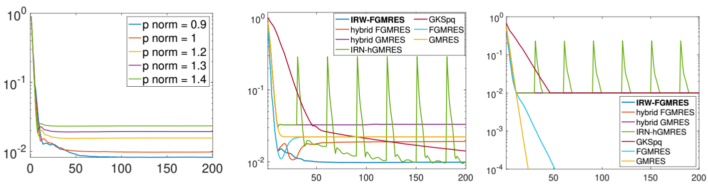


Fig. 2: star_cluster test problem tested with the discrepancy principle. **(a)** history of the relative error norms (i.e., relative error norm against iteration number) for the new IRW-FGMRES and different p norms. **(b)** history of the relative error norms for the new IRW-FGMRES and other Krylov-based solvers, **(c)** history of the relative residual norms (i.e., relative residual norm against iteration number) for the new IRW-FGMRES and other Krylov-based solvers

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